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Origin of the Landau–Lifshitz Hydrodynamic Fluctuations in Nonequilibrium Systems and a New Method for Reducing the Boltzmann Equation

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The Landau–Lifshitz fluctuating fluxes in fluctuating hydrodynamics are derived from the *deterministic* Boltzmann equation with the aid of a reduction method developed by Fujisaka and Mori. Thus it is shown that the hydrodynamic fluctuations in *nonequilibrium systems* are generated by the reduction of variables from the μ-space distribution function to its five momentum moments, i.e., the hydrodynamic variables. This differs from the Bixon–Zwanzig and Fox–Uhlenbeck theories, in which the Landau–Lifshitz fluctuating fluxes are derived from the molecular fluctuating force in the *stochastic* Boltzmann–Langevin equation, which is, however, negligible in nonequilibrium systems. Thus the present method improves the Chapman–Enskog reduction method so as to include the hydrodynamic fluctuations generated by the reduction of variables.

KEY WORDS: Reduction of variables; generation of fluctuations; fluctuating forces; the Boltzmann equation; collisional invariants; hydrodynamic variables; conservation laws; transport fluxes; fluctuating fluxes; the fluctuation-dissipation theorem of the second kind.

1. INTRODUCTION

Hydrodynamic fluctuations play an important role not only in equilibrium systems but also in nonequilibrium systems. They determine the linear responses of equilibrium systems to external disturbances.[1,2] In nonequilibrium systems, they become important in the critical region of hydrodynamic instabilities and contribute to macroscopic behavior.[3,4] A generalized fluid dynamics was proposed by Landau and Lifshitz[2] as a theory including such hydrodynamic fluctuations. In this theory, however,

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fluctuating fluxes are just added to the usual hydrodynamic equations, and the fluctuation-dissipation theorem, which relates their time correlation functions to the transport coefficients, is assumed to hold even in nonequilibrium systems. Recently it has been realized that anomalous fluctuations occur near instability points of the open systems far from thermal equilibrium. Therefore it is important to be able to derive the Landau–Lifshitz hydrodynamic fluctuations in nonequilibrium systems from a basic standpoint. To do this we consider a neutral dilute gas for which the Boltzmann equation holds, since the Boltzmann equation enables us to define and specify nonequilibrium systems most clearly.

The hydrodynamic processes of dilute gases are characterized by space-time coarse-graining with the space-time cutoff (b_h, t_{ch}) satisfying $b_h \gg l_f$, $t_{ch} \gg \tau_f$, where l_f is the mean free path of molecules and τ_f is the mean free time. The hydrodynamic equations for describing these processes can be derived from the kinetic equations for the kinetic processes characterized by (l_f, τ_f). The kinetic equations have been derived rigorously from the mechanical equations of motion by Tokuyama and Mori.[5] According to their theory, the spatially coarse-grained particle density $N_{\mathbf{pr}}(t)$ in μ-space with a length cutoff b satisfying $l_f \gg b \gg r_0$, where r_0 is the molecular diameter, is governed by a stochastic equation of motion:

$$\partial_t N_{\mathbf{pr}}(t) = -\sum_{\alpha} (p_\alpha/m)\, \partial_\alpha N_{\mathbf{pr}}(t) + C_{\mathbf{pr}}(N) + G_{\mathbf{pr}}(t) \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_\alpha \equiv \partial/\partial r_\alpha$, $C_{\mathbf{pr}}(N)$ is the Boltzmann collision term, and $G_{\mathbf{pr}}(t)$ is a molecular fluctuating force. $N_{\mathbf{pr}}$ is decomposed into a deterministic part $F_{\mathbf{pr}}$ and a fluctuating part $Z_{\mathbf{pr}}$ ($N_{\mathbf{pr}} = F_{\mathbf{pr}} + Z_{\mathbf{pr}}$) whose scaling properties differ from each other. In *nonequilibrium systems* with three dimensions, the fluctuating part $Z_{\mathbf{pr}}$ is negligible compared to the deterministic part $F_{\mathbf{pr}}$.[5] Hence (1) leads to the *deterministic* Boltzmann equation

$$\partial_t F_{\mathbf{pr}}(t) = -\sum_{\alpha} (p_\alpha/m)\, \partial_\alpha F_{\mathbf{pr}}(t) + C_{\mathbf{pr}}(F) \tag{2}$$

On the other hand, in *equilibrium systems*, the deterministic part $F_{\mathbf{pr}}$ is the Maxwell distribution $F_{\mathbf{p}}^e$, which is a stationary solution of (2), and $Z_{\mathbf{pr}}$ represents fluctuations around $F_{\mathbf{p}}^e$:

$$N_{\mathbf{pr}}(t) = F_{\mathbf{p}}^e + Z_{\mathbf{pr}}(t) \tag{3}$$

Therefore the deterministic part $F_{\mathbf{pr}}$ does not contribute to the time evolution at all, and the time evolution is determined by the fluctuations $Z_{\mathbf{pr}}(t)$ for which (1) leads to the *stochastic* Boltzmann–Langevin equation[5]

$$\partial_t Z_{\mathbf{pr}}(t) = -\sum_{\alpha} (p_\alpha/m)\, \partial_\alpha Z_{\mathbf{pr}}(t) + D_{\mathbf{p}}^e Z_{\mathbf{pr}}(t) + G_{\mathbf{pr}}(t) \tag{4}$$

where $D_{\mathbf{p}}^{e}$ is the linear collision operator. Thus the molecular fluctuating force $G_{\mathbf{pr}}(t)$ is not negligible, in contrast to the case of nonequilibrium systems.

Attempts to derive the phenomenological Landau–Lifshitz fluctuating fluxes from the kinetic equations have been made by Bixon and Zwanzig[6] and Fox and Uhlenbeck.[7] They derived them from the fluctuating force of the Boltzmann–Langevin equation (4). Therefore, the origin of the hydrodynamic fluctuations is the molecular fluctuations, and their theories are valid only in equilibrium systems. On the other hand, in *nonequilibrium systems*, the situation is quite different, since the fluctuating part is negligible and the deterministic Boltzmann equation (2) must be used. In this case, the Landau–Lifshitz fluctuating fluxes must be derived from the fluctuations generated by the reduction of variables from the μ-space distribution function $F_{\mathbf{pr}}$ to the hydrodynamic variables. Namely, in nonequilibrium systems, the molecular fluctuations are negligible, and the Landau–Lifshitz fluctuating fluxes must be generated by the reduction of variables. This cannot be treated by the Chapman–Enskog reduction method. A new reduction method which can treat such a generation of fluctuations has been developed by Fujisaka and Mori.[8]

In the present paper, we shall study the origin of the hydrodynamic fluctuations and the structure of the transport coefficients with the aid of the Fujisaka–Mori reduction method in both equilibrium and nonequilibrium systems. It will be shown that in nonequilibrium systems, the origin of the hydrodynamic fluctuations is the fluctuations generated by the reduction of variables for $F_{\mathbf{pr}}$, and the fluctuating fluxes are related to the linear transport coefficients through a fluctuation-dissipation theorem of the second kind, whereas in equilibrium systems, we obtain the same results as Bixon and Zwanzig, and Fox and Uhlenbeck.

2. A NEW REDUCTION METHOD

The hydrodynamic variables represent the collisional invariants for which the Boltzmann equation leads to the conservation laws. The collisional invariants are

$$g_\mu(\mathbf{p}) \equiv \begin{pmatrix} 1 \\ p_\alpha \\ p^2/2m \end{pmatrix} \tag{5}$$

where m is the mass and p_α is the α component of momentum. The hydrodynamic variables are defined by

$$A_{\mu\mathbf{r}}(t) \equiv A_{\mu\mathbf{r}}(N(t)) \equiv \int d\mathbf{r}'\, \Delta_h(\mathbf{r} - \mathbf{r}')\left[\int d\mathbf{p}\, g_\mu(\mathbf{p}) N_{\mathbf{pr}'}(t)\right] \tag{6}$$

with $\Delta_h(\mathbf{r})$ the coarse-grained δ-function

$$\Delta_h(\mathbf{r}) \equiv (1/\Omega) \sum_{\mathbf{q}}{}' \exp(-i\mathbf{q}\cdot\mathbf{r}), \qquad q \leq 1/b_h \tag{7}$$

where Ω is the volume of the system and $\sum_{\mathbf{q}}{}'$ is the sum over the wave vectors $\mathbf{q}$ whose magnitudes are smaller than the cutoff $q_c \equiv 1/b_h$. Thus we take the coarse-grained hydrodynamic variables in accordance with the hydrodynamic space-time coarse-graining. They take the form

$$A_{\mu\mathbf{r}}(t) = \begin{pmatrix} A_{0\mathbf{r}}(t) \\ A_{1\alpha,\mathbf{r}}(t) \\ A_{2\mathbf{r}}(t) \end{pmatrix} = \begin{pmatrix} n \\ mnu_\alpha \\ \frac{3}{2}nk_BT + mnu^2/2 \end{pmatrix} \tag{8}$$

where $n(\mathbf{r})$, $u_\alpha(\mathbf{r})$, and $T(\mathbf{r})$ are the coarse-grained local particle density, local fluid velocity, and local temperature, respectively.

The hydrodynamic variables are the first five momentum moments of $N_{\mathbf{pr}}$. Therefore, in order to derive the hydrodynamic equations from the kinetic equation for $N_{\mathbf{pr}}$, we have to eliminate the higher moments other than the hydrodynamic variables. We first discuss a new reduction method which takes into account the fluctuations generated by the reduction of variables correctly.

2.1. General Formula

Let us write the stochastic equation of motion (1) as

$$\partial_t N_i(t) \equiv \zeta_i(N) + G_i(t) \tag{9}$$

and introduce the distribution for $N(t) \equiv \{N_i(t)\}$ to have a set of values $f \equiv \{f_i\}$,

$$\Pi_f(t) \equiv \delta(N(t) - f) \equiv \prod_i \delta(N_i(t) - f_i) \tag{10}$$

As was shown by Tokuyama and Mori,[5] its time evolution is given by

$$\partial_t \Pi_f(t) = M(f)\Pi_f(t) + G_f(t) \tag{11a}$$

$$\Pi_f(t) = e^{tM(f)}\Pi_f(0) + \int_0^t ds\, e^{(t-s)M(f)}G_f(s) \tag{11b}$$

where

$$M(f) \equiv M_0(f) + M_1(f) \equiv -\sum_i \frac{\partial}{\partial f_i} \zeta_i(f) + \sum_i \sum_j \frac{\partial}{\partial f_i}\frac{\partial}{\partial f_j} E_{ij}(f) \tag{12}$$

$$G_i(t) = \int f_i G_f(t)\, df \tag{13}$$

In (12), $M_0(f)$ represents the drift term and $M_1(f)$ the diffusion term. The fluctuating forces satisfy the orthogonality condition

$$\langle G_i(t)H(N(0))\rangle = \langle G_f(t)H(N(0))\rangle = 0 \tag{14}$$

where H is an arbitrary functional of $N(0)$, and a fluctuation-dissipation theorem of the second kind leads to

$$\langle G_i(t_1)G_j(t_2);f\rangle = 2\delta(t_1 - t_2)E_{ij}(f) \tag{15}$$

The time evolution of any functional of $N(t)$, $B(N(t))$, can be written as

$$B(t) = \int df\, B(f)\Pi_f(t) = \int df\left[\hat{B}(t)\Pi_f(0) + \int_0^t ds\, \hat{B}(t-s)G_f(s)\right] \tag{16a}$$

$$\partial_t B(t) = \int df\left[\Pi_f(0)e^{tM^+} + \int_0^t ds\, G_f(s)e^{(t-s)M^+}\right]M^+B(f) + \int df\, B(f)G_f(t) \tag{16b}$$

where

$$\hat{B}(t) \equiv \exp(tM^+)\, B(f) \tag{17}$$

$$M^+(f) \equiv M_0^+(f) + M_1^+(f) \equiv \sum_i \zeta_i(f)\frac{\partial}{\partial f_i} + \sum_i\sum_j E_{ij}(f)\frac{\partial}{\partial f_j}\frac{\partial}{\partial f_i} \tag{18}$$

Using the Mori identity for linear operators L, P, and $Q \equiv 1 - P$,

$$e^{tL} = e^{tL}P + \int_0^t ds\, e^{(t-s)L}PLe^{sQL}Q + e^{tQL}Q \tag{19}$$

we can rewrite (16b) as

$$\partial_t B(t) = \int df\, \Pi_f(t)[PM^+B(f)] + \int_0^t ds \int df\, \Pi_f(t-s)[PM^+\hat{Q}_B(s)] + F_B(t) \tag{20}$$

where

$$\hat{Q}_B(t) \equiv \exp[t(1-P)M^+]\,(1-P)M^+B(f) \tag{21}$$

$$F_B(t) \equiv Q_B(t) + \int df\, B(f)G_f(t) + \int_0^t ds \int df\, \hat{Q}_B(t-s)G_f(s) \tag{22}$$

$$Q_B(t) \equiv [\hat{Q}_B(t)]_{f\to N(0)} \tag{23}$$

Let us take as P a projector onto the hydrodynamic variables $A(f) \equiv \{A_{\mu\mathbf{r}}(f)\}$,

$$PB(f) \equiv \langle B(f); A(f) \rangle \tag{24}$$

where

$$\langle B(f); a \rangle \equiv \int df\, B(f) p_a(f) \tag{25}$$

$$p_a(f) \equiv \rho_0(f)\, \delta(A(f) - a)/w(a) \tag{26}$$

$$w(a) \equiv \int df\, \rho_0(f)\, \delta(A(f) - a) \tag{27}$$

Here $\rho_0(f)$ is the initial distribution for $N(0)$, and $p_a(f)$ is the conditional probability distribution with the value of $A(f)$ being fixed so as to be a. Then (20) takes the form

$$\partial_t B(t) = \langle M^+(f) B(f); A(t) \rangle + \int_0^t ds\, \langle M^+(f) \hat{Q}_B(s); A(t-s) \rangle + F_B(t) \tag{28}$$

We take as $B(t)$ the moment generator of the hydrodynamic variables,

$$\Pi_a(t) \equiv \delta(A(t) - a) = \int df\, \delta(A(f) - a) \Pi_f(t) \tag{29}$$

Then (28) leads to a hydrodynamic master equation

$$\partial_t \Pi_a(t) = \langle M^+(f)\, \delta(A(f) - a); A(t) \rangle + \int_0^t ds\, \langle M^+(f) \hat{Q}_a(s); A(t-s) \rangle + F_a(t) \tag{30}$$

where $\hat{Q}_a(t)$ and $F_a(t)$ are given by (21) and (22) with $\delta(A(f) - a)$ for $B(f)$. This is a special case of the reduced equation formulated by Fujisaka and Mori.[8] It is worth noting that the master fluctuating force $F_a(t)$ is composed of three parts: the fluctuating force $Q_a(t)$ generated by the reduction of variables, the molecular part $G_f(t)$, and their cross term. The first-moment equation of this master equation leads to a reduced equation of motion for $A_{\mu\mathbf{r}}(t)$ which gives the fluctuating hydrodynamic equations.

The initial distribution $\rho_0(f)$ must be chosen so as to give the statistical ensemble of the system correctly before the lapse of the hydrodynamic time scale τ_h. In the usual nonequilibrium fluids even in turbulent states, the local states are approximately in thermal equilibrium with the local density $n(\mathbf{r})$, local fluid velocity $u_\alpha(\mathbf{r})$, and local temperature $T(\mathbf{r})$, and the local equilibrium ensemble produces the correct nonequilibrium ensemble in a few mean free times. Then we can take the local equilibrium distribution as $\rho_0(f)$.

It is worth noting here a physical meaning of the reduction of variables. The hydrodynamic variables $A(t)$ are the only slowly varying degrees of freedom of the system in the hydrodynamic length and time scale. They are, however, coupled to the rapidly varying degrees of freedom whose length scales are smaller than b_h or whose time scales are shorter than t_{ch}. In order to take into account this coupling fully, we eliminate the rapidly varying degrees of freedom and derive a closed equation of motion for the hydrodynamic variables $A(t)$. Equation (30) is such a closed equation. Thus the hydrodynamic equations are derived and the dissipative terms represent the renormalization by the rapidly varying degrees of freedom.

2.2. Nonequilibrium Systems

Let us start with the stochastic equation of motion (1), which leads to

$$\zeta_i(f) = \zeta_{\mathbf{pr}}(f) = -\sum_\alpha \frac{p_\alpha}{m} \partial_\alpha f_{\mathbf{pr}} + C_{\mathbf{pr}}(f) \tag{31}$$

$$M_0(f) = -\iint \frac{d\mathbf{p}\, d\mathbf{r}}{\omega} \frac{\partial}{\partial f_{\mathbf{pr}}} \left[-\sum_\alpha \frac{p_\alpha}{m} \partial_\alpha f_{\mathbf{pr}} + C_{\mathbf{pr}}(f) \right] \tag{32}$$

$$M_1(f) = \iint \frac{d\mathbf{p}\, d\mathbf{r}}{\omega} \iint \frac{d\mathbf{p}'\, d\mathbf{r}'}{\omega} \frac{\partial}{\partial f_{\mathbf{pr}}} \frac{\partial}{\partial f_{\mathbf{p}'\mathbf{r}'}} E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(f) \tag{33}$$

where ω is the phase volume of the coarse-graining cell in μ-space.[5] In nonequilibrium systems, let us take, as the initial distribution, the local equilibrium distribution

$$\rho_0(f) = N(a) \exp\left[-(1/2) \iint d\mathbf{p}\, d\mathbf{r}\, (\delta f_{\mathbf{pr}})^2 / F^l_{\mathbf{pr}}(a) \right] \tag{34}$$

where

$$\delta f_{\mathbf{pr}} \equiv f_{\mathbf{pr}} - F^l_{\mathbf{pr}}(a) \tag{35}$$

$$[F^l_{\mathbf{pr}}(a)]_{a \to A(t)} = F^l_{\mathbf{pr}}(t) \equiv [n/(2\pi m k_B T)^{3/2}] \exp[-(\mathbf{p} - m\mathbf{u})^2 / 2mk_B T] \tag{36}$$

Using

$$M_0{}^+ \, \delta(A(f) - a) = -\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \frac{\partial}{\partial a_{\mu\mathbf{r}}} \{[M_0{}^+ A_{\mu\mathbf{r}}(f)]\, \delta(A(f) - a)\}$$

and taking $M\rho_0(f) \doteqdot 0$ in the dissipative second term, we can transform (30) into

$$\partial_t \Pi_a(t) \doteqdot -\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \frac{\partial}{\partial a_{\mu\mathbf{r}}} [v_{\mu\mathbf{r}}(a)\Pi_a(t)] - \int_0^t ds \int db\, \Pi_b(t-s) \frac{\langle \hat{Q}_a(s)\hat{Q}_b'(0)\rangle}{w(b)} + F_a(t) \tag{37}$$

with

$$v_{\mu\mathbf{r}}(a) \equiv \langle M_0{}^+ A_{\mu\mathbf{r}}(f); a\rangle - \sum_\nu \int \frac{d\mathbf{r}'}{\omega_h} \frac{\partial}{\partial a_{\mu\mathbf{r}'}} \iint d\mathbf{p}\, d\mathbf{p}'\, g_\mu(\mathbf{p}) g_\nu(\mathbf{p}') \times \langle E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(f); a\rangle \tag{38}$$

$$\hat{Q}_a(t) \equiv \exp[t(1-P)M^+]\,(1-P)M^+\, \delta(A(f)-a) \tag{39a}$$

$$\hat{Q}_b'(0) \equiv \hat{Q}_b(0) - 2\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \iiint d\mathbf{p}\, d\mathbf{p}'\, d\mathbf{r}' \frac{\partial}{\partial b_{\mu\mathbf{r}}} \delta(A(f)-b) g_\mu(\mathbf{p}) \times \left[\frac{1}{\omega}\frac{\partial}{\partial f_{\mathbf{p}'\mathbf{r}'}} - \frac{\partial f_{\mathbf{p}'\mathbf{r}'}}{F^l_{\mathbf{p}'\mathbf{r}'}}\right] E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(f) \tag{39b}$$

$$F_a(t) = Q_a(t) + \int df\, \delta(A(f)-a) G_f(t) + \int_0^t ds \int df\, \hat{Q}_a(t-s) G_f(s) \tag{40}$$

where use has been made of $(1/\omega)\,\partial f_{\mathbf{p}'\mathbf{r}'}/\partial f_{\mathbf{pr}} = \delta(\mathbf{p}-\mathbf{p}')\,\Delta(\mathbf{r}-\mathbf{r}')$ and ω_h is the volume of the hydrodynamic coarse-graining cell, $\omega_h \equiv (b_h)^d$, with d the spatial dimensionality. Equation (37) has the standard structure of the master equation in the memory function formalism.[9]

In nonequilibrium systems, however, the fluctuating part $Z_{\mathbf{pr}}(t)$ is negligible compared to the deterministic part $F_{\mathbf{pr}}(t)$. Therefore the molecular fluctuating forces $G_{\mathbf{pr}}(t)$ and $G_f(t)$ and the diffusion coefficients $E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}$ can also be neglected. Then the starting equation becomes the deterministic Boltzmann equation (2), and the corresponding master equation reduces to

$$\partial_t \Pi_f(t) = M_0(f)\Pi_f(t), \qquad \Pi_f(t) \equiv \delta(F(t)-f) \tag{41}$$

Therefore (37) reduces to

$$\partial_t \Pi_a(t) = -\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \frac{\partial}{\partial a_{\mu\mathbf{r}}} [v^0_{\mu\mathbf{r}}(a)\Pi_a(t)] - \int_0^t ds \int db\, \Pi_b(t-s) \frac{\langle \hat{Q}_a{}^0(s)\hat{Q}_b{}^0(0)\rangle}{w(b)} + F_a{}^0(t) \tag{42}$$

with

$$v_{\mu\mathbf{r}}^0(a) \equiv \langle M_0{}^+ A_{\mu\mathbf{r}}(f); a \rangle \tag{43}$$

$$\hat{Q}_a{}^0(t) \equiv -\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \frac{\partial}{\partial a_{\mu\mathbf{r}}} [U(t)\hat{R}_{\mu\mathbf{r}}(0)\, \delta(A(f) - a)] \tag{44}$$

$$F_a{}^0(t) \equiv Q_a{}^0(t) \tag{45}$$

where

$$U(t) \equiv \exp[t(1 - P)M_0{}^+] \tag{46}$$

$$\hat{R}_{\mu\mathbf{r}}(t) \equiv U(t)(1 - P)M_0{}^+ A_{\mu\mathbf{r}}(f) \tag{47}$$

The reduced equation of motion for $A(t)$ is given by

$$\partial_t A_{\mu\mathbf{r}}(t) = v_{\mu\mathbf{r}}^0(A(t)) + \int_0^t ds \Bigg[\sum_\nu \int \frac{d\mathbf{r}'}{\omega_h} \frac{1}{w(a)} \frac{\partial}{\partial a_{\nu\mathbf{r}'}} \times \{w(a)\langle \hat{R}_{\mu\mathbf{r}}(s)\hat{R}_{\nu\mathbf{r}'}(0); a\rangle\}\Bigg]_{a = A(t-s)} + R_{\mu\mathbf{r}}(t) \tag{48}$$

where use has been made of $\partial a_{\mu\mathbf{r}}/\partial a_{\mu'\mathbf{r}'} = \omega_h\, \delta_{\mu,\mu'}\, \Delta_h(\mathbf{r} - \mathbf{r}')$ and

$$R_{\mu\mathbf{r}}(t) \equiv [\hat{R}_{\mu\mathbf{r}}(t)]_{f \to F(0)} \tag{49}$$

This has the standard structure of the reduced equation of motion in the memory function formalism.[9] Thus, in nonequilibrium systems, the propagator is given by the drift term $M_0{}^+(f)$, which is determined by the Boltzmann equation (2), and the hydrodynamic fluctuating forces $R_{\mu\mathbf{r}}(t)$ are generated by the reduction of variables from $F_{\mathbf{pr}}(t)$ to $A_{\mu\mathbf{r}}(t)$.

2.3. Equilibrium Systems

In equilibrium systems, the deterministic part $F_{\mathbf{pr}}$ is the Maxwell distribution, and the hydrodynamic variables (6) are written as

$$A_{\mu\mathbf{r}}(t) = A_{\mu\mathbf{r}}(Z(t)) = \int d\mathbf{r}'\, \Delta_h(\mathbf{r} - \mathbf{r}')\left\{\int d\mathbf{p}\, g_\mu(\mathbf{p})[F_p{}^e + Z_{\mathbf{pr}'}(t)]\right\} \tag{50}$$

Namely, the $A_{\mu\mathbf{r}}(t)$ are the functionals of $Z_{\mathbf{pr}}(t)$, and the generator (29) can be written as

$$\Pi_a(t) = \delta(A(t) - a) = \int dz\, \delta(A(z) - a)\Pi_z(t) \tag{51}$$

where

$$\Pi_z(t) \equiv \delta(Z(t) - z) \tag{52}$$

Therefore, we start with the Boltzmann–Langevin equation (4) and take, as the initial distribution $\rho_0(f)$, the equilibrium distribution

$$\rho_e(z) = N(\infty)\exp\left[-(1/2)\iint d\mathbf{p}\, d\mathbf{r}\, (z_{\mathbf{pr}})^2/F_{\mathbf{p}}{}^e\right] \tag{53}$$

Then, as will be shown in Appendix A, (30) leads to

$$\partial_t\Pi_a(t) = -\sum_\mu \int \frac{d\mathbf{r}}{\omega_h}\frac{\partial}{\partial a_{\mu\mathbf{r}}}[v^0_{\mu\mathbf{r}}(a)\Pi_a(t)] - \int_0^t ds \int db\, \Pi_b(t-s)\frac{\langle \hat{Q}_a{}^0(s)\hat{Q}_b{}^0(0)\rangle}{w(b)} + F_a(t) \tag{54}$$

$$\partial_t A_{\mu\mathbf{r}}(t) = v^0_{\mu\mathbf{r}}(A(t)) + \int_0^t ds \left\{\sum_\nu \int \frac{d\mathbf{r}'}{\omega_h}\frac{1}{w(a)}\frac{\partial}{\partial a_{\nu\mathbf{r}'}} \times [w(a)\langle \hat{R}_{\mu\mathbf{r}}(s)\hat{R}_{\nu\mathbf{r}'}(0); a\rangle]\right\}_{a=A(t-s)} + R'_{\mu\mathbf{r}}(t) \tag{55}$$

with

$$F_a(t) = Q_a{}^0(t) + \int_0^t ds \int dz\, \hat{Q}_a{}^0(t-s)G_z(s) + \int dz\, \delta(A(z)-a)G_z(t) \tag{56}$$

$$R'_{\mu\mathbf{r}}(t) = R_{\mu\mathbf{r}}(t) + \int_0^t ds \int dz\, \hat{R}_{\mu\mathbf{r}}(t-s)G_z(s) \tag{57}$$

The systematic parts of these equations are identical to those of (42) and (48). However, the fluctuating forces (56) and (57) differ from the fluctuating forces $Q_a{}^0(t)$ and $R_{\mu\mathbf{r}}(t)$.

3. HYDRODYNAMIC FLUCTUATIONS IN THE NONEQUILIBRIUM SYSTEMS

In the linear dissipative case, a linear relation between fluxes and forces holds. Then the reduced equations of motion (48) lead to fluctuating hydrodynamic equations with the linear transport coefficients. Before treating this case, however, we investigate the structure of (48). With the aid of the relation[5]

$$(1/\omega)\,\partial f_{\mathbf{p}'\mathbf{r}'}/\partial f_{\mathbf{pr}} = \delta(\mathbf{p}'-\mathbf{p})\,\Delta(\mathbf{r}'-\mathbf{r}) \tag{58}$$

we find that (6) and (32) lead to

$$M_0{}^+(f)A_{\mu\mathbf{r}}(f) = -\sum_\alpha \partial_\alpha j^\alpha_{\mu\mathbf{r}}(f) \tag{59}$$

where

$$j_{\mu\mathbf{r}}^{\alpha}(f) \equiv \int d\mathbf{r}'\, \Delta_h(\mathbf{r}-\mathbf{r}')\left[\int d\mathbf{p}\,(p_\alpha/m) g_\mu(\mathbf{p}) f_{\mathbf{p}\mathbf{r}'}\right] \tag{60}$$

and the collision term of $M_0{}^+$ does not contribute since the $A_{\mu\mathbf{r}}(f)$ are the collisional invariants. Therefore (43) and (47) lead to

$$v_{\mu\mathbf{r}}^{0}(a) = -\sum_\alpha \partial_\alpha[n(\mathbf{r}) h_{\mu\mathbf{r}}^{\alpha}(a)] \tag{61}$$

$$\hat{R}_{\mu\mathbf{r}}(t) = -\sum_\alpha \partial_\alpha \hat{S}_{\mu\mathbf{r}}^{\alpha}(t) \tag{62}$$

where

$$h_{\mu\mathbf{r}}^{\alpha}(a) \equiv \langle j_{\mu\mathbf{r}}^{\alpha}(f); a\rangle / n(\mathbf{r}) \tag{63}$$

$$\hat{S}_{\mu\mathbf{r}}^{\alpha}(t) \equiv U(t)(1-P) j_{\mu\mathbf{r}}^{\alpha}(f) \tag{64}$$

Since (24) and (34) lead to $Pf_{\mathbf{pr}} = F_{\mathbf{pr}}^{l}(a)$, (63) and (64) reduce to

$$h_{\mu\mathbf{r}}^{\alpha}(a) = \begin{pmatrix} u_\alpha \\ m u_\alpha u_\beta + k_B T\, \delta_{\alpha,\beta} \\ (mu^2/2 + 5k_B T/2) u_\alpha \end{pmatrix} \tag{65}$$

$$\hat{S}_{\mu\mathbf{r}}^{\alpha}(t) = \int d\mathbf{r}'\, \Delta_h(\mathbf{r}-\mathbf{r}')\left[\int d\mathbf{p}\, J_{\mu\mathbf{r}}^{\alpha}(\mathbf{p}) U(t)\, \delta f_{\mathbf{p}\mathbf{r}'}\right] \tag{66}$$

with

$$J_{\mu\mathbf{r}}^{\alpha}(\mathbf{p}) \equiv \begin{pmatrix} 0 \\ J_{1\mathbf{r}}^{\alpha\beta}(\mathbf{p}) \\ J_{2\mathbf{r}}^{\alpha}(\mathbf{p}) + \sum_\beta u_\beta(\mathbf{r}) J_{1\mathbf{r}}^{\alpha\beta}(\mathbf{p}) \end{pmatrix} \tag{67}$$

Here we have defined the kinetic transport fluxes

$$J_{1\mathbf{r}}^{\alpha\beta}(\mathbf{p}) \equiv m(V_\alpha V_\beta - V^2\, \delta_{\alpha,\beta}/3) \tag{68}$$

$$J_{2\mathbf{r}}^{\alpha}(\mathbf{p}) \equiv (mV^2/2 - 5k_B T/2) V_\alpha \tag{69}$$

with V_α the thermal velocity, $V_\alpha \equiv (p_\alpha/m) - u_\alpha$.

The hydrodynamic variables satisfy the laws of conservation of mass, momentum, and energy

$$\partial_t A_{\mu\mathbf{r}}(t) = -\sum_\alpha \partial_\alpha j_{\mu\mathbf{r}}^{\alpha}(t) \tag{70}$$

In fact, the reduced equation of motion (48) takes this conservation form with the flux densities

$$j_{\mu\mathbf{r}}^{\alpha}(t) = n(\mathbf{r}, t) h_{\mu\mathbf{r}}^{\alpha}(A(t)) + \mathscr{T}_{\mu\mathbf{r}}^{\alpha}(t) \tag{71}$$

where we have defined the transport fluxes

$$\mathscr{T}^{\alpha}_{\mu\mathbf{r}}(t) \equiv \sum_{\nu}\sum_{\beta}\int_0^t ds \int d\mathbf{r}' \left\{\left[\frac{1}{k_B} X^{\beta}_{\nu\mathbf{r}'}(a) - \frac{1}{\omega_h}\frac{\partial}{\partial a_{\nu\mathbf{r}'}}\partial_{\beta}'\right] \times \langle \hat{S}^{\alpha}_{\mu\mathbf{r}}(s)\hat{S}^{\beta}_{\nu\mathbf{r}'}(0); a\rangle\right\}_{a=A(t-s)} + S^{\alpha}_{\mu\mathbf{r}}(t) \tag{72}$$

$$S^{\alpha}_{\mu\mathbf{r}}(t) \equiv [\hat{S}^{\alpha}_{\mu\mathbf{r}}(t)]_{f\to F(0)} \tag{73}$$

Here the $X^{\beta}_{\nu\mathbf{r}}(a)$ are the thermodynamic forces

$$X^{\beta}_{\nu\mathbf{r}}(a) = \partial_{\beta}[(1/\omega_h)(\partial/\partial a_{\nu\mathbf{r}})\sigma(a)] \tag{74}$$

with $\sigma(a) \equiv k_B \log w(a)$ the entropy of the thermodynamic state a.

The $S^{\alpha}_{\mu\mathbf{r}}(t)$ in (72) represent the fluctuating part of the flux densities, and lead to the Landau–Lifshitz fluctuating fluxes. Since the length cutoff b_h of the r dependence of (66) is much longer than the mean free path of molecules, we have, on the hydrodynamic space-time scale,

$$\langle \hat{S}^{\alpha}_{\mu\mathbf{r}}(t)\hat{S}^{\beta}_{\nu\mathbf{r}'}(0); a\rangle = 2k_B L_{\mu\alpha;\nu\beta}(a_{\mathbf{r}})\,\Delta_h(\mathbf{r}-\mathbf{r}')\,\Delta_h(t) \tag{75}$$

$$L_{\mu\alpha;\nu\beta}(a_{\mathbf{r}}) \equiv (1/k_B)\int_0^{\infty} dt\,\langle \hat{S}^{\alpha}_{\mu\mathbf{r}}(t)\hat{S}^{\beta}_{\nu\mathbf{r}}(0); a_{\mathbf{r}}\rangle \tag{76}$$

where $a_{\mathbf{r}}$ denotes the hydrodynamic variables at $\mathbf{r}$. As will be shown in Appendix B, (66) can be written in (76) as

$$\hat{S}^{\alpha}_{\mu\mathbf{r}}(t) \cong \int d\mathbf{r}'\,\Delta_h(\mathbf{r}-\mathbf{r}')\int d\mathbf{p}\,[\exp(t\tilde{D}_{\mathbf{pr}})\,J^{\alpha}_{\mu\mathbf{r}}(\mathbf{p})]\,\delta f_{\mathbf{pr}'} \tag{77}$$

where $\tilde{D}_{\mathbf{pr}}$ is defined by

$$\tilde{D}_{\mathbf{pr}}\psi \equiv (1/F^{l}_{\mathbf{pr}})D_{\mathbf{pr}}[F^{l}_{\mathbf{pr}}\psi] \tag{78}$$

with $D_{\mathbf{pr}}$ the linear collision operator in the local equilibrium state. Hence, inserting (77) into (76) and using $\langle \delta f_{\mathbf{pr}}\,\delta f_{\mathbf{p}'\mathbf{r}'}; a\rangle = \delta(\mathbf{p}-\mathbf{p}')\,\Delta(\mathbf{r}-\mathbf{r}')F^{l}_{\mathbf{pr}}(a)$, we obtain

$$L_{\mu\alpha;\nu\beta}(a_{\mathbf{r}}) = (1/k_B)\int_0^{\infty} dt \int d\mathbf{p}\,[\exp(t\tilde{D}_{\mathbf{pr}})\,J^{\alpha}_{\mu\mathbf{r}}(\mathbf{p})]J^{\beta}_{\nu\mathbf{r}}(\mathbf{p})F^{l}_{\mathbf{pr}}(a) \tag{79}$$

Using (75) in (72) leads to

$$\mathscr{T}^{\alpha}_{\mu\mathbf{r}}(t) = \sum_{\nu}\sum_{\beta}\left\{\left[X^{\beta}_{\nu\mathbf{r}}(a_{\mathbf{r}}) - \frac{k_B}{\omega_h}\frac{\partial}{\partial a_{\nu\mathbf{r}}}\partial_{\beta}\right]L_{\mu\alpha;\nu\beta}(a_{\mathbf{r}})\right\}_{a\to A(t)} + S^{\alpha}_{\mu\mathbf{r}}(t) \tag{80}$$

The $L_{\mu\alpha;\nu\beta}(a_{\mathbf{r}})$ represent the transport coefficients. Equation (75) gives a generalized fluctuation-dissipation theorem of the second kind.

In linear dissipative systems, the a dependence of (79) is negligible. Then (80) and (79) reduce to

$$\mathscr{T}^{\alpha}_{\mu\mathbf{r}}(t) \cong \sum_{\nu}\sum_{\beta} L_{\mu\alpha;\nu\beta} X^{\beta}_{\nu\mathbf{r}}(A(t)) + S^{\alpha}_{\mu\mathbf{r}}(t) \tag{81a}$$

$$L_{\mu\alpha;\nu\beta} \equiv (1/k_B)\int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^{e})\, J^{\beta}_{\mu\mathbf{r}}(\mathbf{p})] J^{\beta}_{\nu\mathbf{r}}(\mathbf{p}) F_{\mathbf{p}}{}^{e} \tag{81b}$$

This is the linear relation between the transport fluxes and the thermodynamic forces. As will be shown in Appendix C, the $L_{\mu\alpha;\nu\beta}$ lead to the shear viscosity η and the thermal conductivity κ:

$$\eta = (1/k_B T)\int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^{e})\, J^{xy}_{1\mathbf{r}}] J^{xy}_{1\mathbf{r}} F_{\mathbf{p}}{}^{e} \tag{82}$$

$$\kappa = (1/k_B T^2)\int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^{e})\, J^{x}_{2\mathbf{r}}] J^{x}_{2\mathbf{r}} F_{\mathbf{p}}{}^{e} \tag{83}$$

Furthermore, as also will be shown in Appendix C, the conservation equations (70) with the flux densities (71) lead to fluctuating hydrodynamic equations which agree with those proposed by Landau and Lifshitz.[2]

4. HYDRODYNAMIC FLUCTUATIONS IN EQUILIBRIUM SYSTEMS

In equilibrium systems, the hydrodynamic fluctuating fluxes and the transport coefficients are derived from the Boltzmann–Langevin equation (4) in the same manner as in the preceding sections. Since (4) is linear and can be integrated to give

$$Z_{\mathbf{pr}}(t) = [\exp(tL_{\mathbf{pr}})] Z_{\mathbf{pr}}(0) + \int_0^t ds\, \{\exp[(t-s)L_{\mathbf{pr}}]\} G_{\mathbf{pr}}(s) \tag{84}$$

with

$$L_{\mathbf{pr}} \equiv -\sum_{\alpha} (p_\alpha/m)\, \partial_\alpha + D_{\mathbf{p}}{}^{e} \tag{85}$$

however, it is simpler to use an L-type projector for the reduction of variables,[10] as will be shown in Appendix D. For a sufficiently large t ($\gg \tau_h$), the first term of (84) vanishes and the time evolution of the hydrodynamic variables (50) is completely governed by the molecular fluctuating force $G_{\mathbf{pr}}(t)$. In the following, we use the notations defined in Appendix D, since this treatment is more convenient for a comparison with Bixon and Zwanzig's theory. Thus we obtain the conservation equations (70) with the transport fluxes

$$\mathscr{T}^{\alpha}_{\mu\mathbf{r}}(t) = \sum_{\nu}\sum_{\beta} L_{\mu\alpha;\nu\beta} X^{\beta}_{\nu\mathbf{r}}(t) + T^{\alpha}_{\mu\mathbf{r}}(t) \tag{86}$$

where

$$L_{\mu\alpha;\nu\beta} \equiv (1/k_B)\int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^e)\, J_\mu{}^\alpha(\mathbf{p})] J_\nu{}^\beta(\mathbf{p}) F_{\mathbf{p}}{}^e \tag{87}$$

The fluctuating fluxes consist of two parts of different origin: $T^\alpha_{\mu\mathbf{r}} = T^\alpha_{1\mu\mathbf{r}} + T^\alpha_{2\mu\mathbf{r}}$,

$$T^\alpha_{1\mu\mathbf{r}}(t) = \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^e)\, J_\mu{}^\alpha(\mathbf{p})] Z_{\mathbf{pr}}(0) \tag{88}$$

$$T^\alpha_{2\mu\mathbf{r}}(t) = \int_0^t ds \int d\mathbf{p}\, [\exp(s\tilde{D}_{\mathbf{p}}{}^e)\, J_\mu{}^\alpha(\mathbf{p})] G_{\mathbf{pr}}(t - s) \tag{89}$$

The first part (88) represents the fluctuating force generated by the reduction of variables, and the second part (89) is the coupling between the first and the molecular fluctuating force $G_{\mathbf{pr}}(t)$. Furthermore, the following relation holds between the transport coefficients and the hydrodynamic fluctuating fluxes:

$$k_B L_{\mu\alpha;\nu\beta} = \int_0^\infty d\tau\, \langle T^\alpha_{\mu\mathbf{r}}(t_0) T^\beta_{\nu\mathbf{r}}(t_0 + \tau)\rangle_e \tag{90}$$

where $\langle \cdots \rangle_e$ denotes the average over the equilibrium ensemble. This equation represents the usual fluctuation-dissipation theorem of the second kind.

Let us consider a sufficiently large time t ($\gg \tau_h$). In this limit, T_1 vanishes and only T_2 due to the molecular fluctuations survives, giving the hydrodynamic fluctuating forces. This situation is the same as that of Bixon and Zwanzig's theory.[6] In this situation, (86) and (90) reduce to

$$\mathscr{T}^\alpha_{\mu\mathbf{r}}(t) = \sum_\nu \sum_\beta L_{\mu\alpha;\nu\beta} X^\beta_{\nu\mathbf{r}}(t) + T^\alpha_{2\mu\mathbf{r}}(t) \tag{91}$$

$$k_B L_{\mu\alpha;\nu\beta} = \int_0^\infty d\tau\, \langle T^\alpha_{2\mu\mathbf{r}}(t_0) T^\beta_{2\nu\mathbf{r}}(t_0 + \tau)\rangle_e \tag{92}$$

Thus, the fluctuating fluxes and the transport coefficients are determined by the molecular fluctuating force $G_{\mathbf{pr}}(t)$.

5. SUMMARY AND REMARKS

It has been shown that the Landau–Lifshitz fluctuating fluxes can be derived from the *deterministic* Boltzmann equation (2) by a statistical reduction of variables. Hence it turns out that Landau–Lifshitz fluctuating hydrodynamics holds even for open systems far from thermal equilibrium, as far as the local states within the spatial cell with volume $b_h{}^d$ ($l_h \gg b_h \gg l_f$) are approximately in local thermal equilibrium. On the other hand, in equilibrium systems, the hydrodynamic fluctuations come out from the molecular fluctuating force $G_{\mathbf{pr}}(t)$ of (4), as was first shown by Bixon and Zwanzig[6] and Fox and Uhlenbeck.[7]

We have developed a new reduction method which takes into account the fluctuations generated by the reduction of variables. Equations (82) and (83) for the transport coefficients, however, lead to the same results as the Chapman–Enskog reduction method.[11] The time integrals lead to the form

$$\left\langle \frac{1}{\tilde{D}_{\mathbf{p}}{}^{e}} \right\rangle \equiv \left(J^{\alpha}_{\mu\mathbf{r}}, \frac{1}{\tilde{D}_{\mathbf{p}}{}^{e}} J^{\alpha}_{\mu\mathbf{r}} \right) \Big/ (J^{\alpha}_{\mu\mathbf{r}}, J^{\alpha}_{\mu\mathbf{r}}) \tag{93}$$

where the parentheses denote the scalar product (A16). A simple approximation is obtained by replacing (93) by $1/\langle \tilde{D}_{\mathbf{p}}{}^{e} \rangle$, which leads to[12]

$$\eta \approx (J^{xy}_{1\mathbf{r}}, J^{xy}_{1\mathbf{r}})/\lambda_1 k_B T, \qquad \kappa \approx (J^{x}_{2\mathbf{r}}, J^{x}_{2\mathbf{r}})/\lambda_2 k_B T^2 \tag{94}$$

where

$$\lambda_1 \equiv -\frac{(J^{xy}_{1\mathbf{r}}, \tilde{D}_{\mathbf{p}}{}^{e} J^{xy}_{1\mathbf{r}})}{(J^{xy}_{1\mathbf{r}}, J^{xy}_{1\mathbf{r}})}, \qquad \lambda_2 \equiv -\frac{(J^{x}_{2\mathbf{r}}, \tilde{D}_{\mathbf{p}}{}^{e} J^{x}_{2\mathbf{r}})}{(J^{x}_{2\mathbf{r}}, J^{x}_{2\mathbf{r}})} \tag{95}$$

It is worth noting that the λ_μ approximately represent the lowest two nonzero eigenvalues of $\tilde{D}_{\mathbf{p}}{}^{e}$, and the $J^{\alpha}_{\mu\mathbf{r}}$ the corresponding eigenfunctions. The calculation of (95) is straightforward, and (94) leads to results identical to the first approximation of Chapman and Enskog.[12] The second approximation can also be obtained from (93) with the aid of a variational principle for $\tilde{D}_{\mathbf{p}}{}^{e}$.

We have started with the stochastic equation (9) for $N_{\mathbf{pr}}(t)$, although the fluctuating part $Z_{\mathbf{pr}}$ and the molecular fluctuating force $G_{\mathbf{pr}}$ are finally neglected in nonequilibrium systems. The reason for this is the following. Since $G_{\mathbf{pr}}(t)$ gives the diffusion coefficient $E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(f)$ and the diffusion term $M_1(f)$, it is needed in order for $M_1(f)$ to ensure that the local equilibrium distribution (34) holds approximately. In this sense, the molecular fluctuating force is also important in nonequilibrium systems.

Equations (81a) and (86) have the same systematic part with identical transport coefficients, although their hydrodynamic fluctuating fluxes $S^{\alpha}_{\mu\mathbf{r}}(t)$ and $T^{\alpha}_{2\mu\mathbf{r}}(t)$ differ notably from each other. This situation may be understood in the following way. Equation (77) indicates that the fluctuating flux $S^{\alpha}_{\mu\mathbf{r}}(t)$ in nonequilibrium systems arises from the molecular collisions represented by $\tilde{D}_{\mathbf{pr}}$. The molecular fluctuating force $G_{\mathbf{pr}}(t)$ and its spectral intensity $E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(F)$, which produce the equilibrium hydrodynamic fluctuating force $T^{\alpha}_{2\mu\mathbf{r}}(t)$, also come out from the molecular collisions. The coupling of the molecular collisions with the hydrodynamic processes is represented by the systematic part of (81a) and (86), namely, by the kinetic transport fluxes $J^{\alpha}_{\mu\mathbf{r}}(\mathbf{p})$, which are approximately the eigenfunctions of $\tilde{D}_{\mathbf{pr}}$. This leads to identical transport coefficients in both equilibrium and nonequilibrium systems.

A generalization of the present theory to dense gases and liquids would be possible with the aid of a theory of generalized Brownian motions[9] and a scaling method for space-time coarse-graining,[4] as far as the fluids are approximately in local thermal equilibrium. This problem and the asymptotic form of (42) will be studied elsewhere.

APPENDIX A. DERIVATION OF (54) AND (55)

For equilibrium systems, let us start from the linear Boltzmann–Langevin equation (4) and the corresponding master equation for the generating function (52). Using the same procedure as in Section 2.2, we obtain from (30) the following master equation:

$$\partial_t \Pi_a(t) = -\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \frac{\partial}{\partial a_{\mu\mathbf{r}}} [v_{\mu\mathbf{r}}(a)\Pi_a(t)] - \int_0^t ds \int db\, \Pi_b(t-s) \frac{\langle \hat{Q}_a(s)\hat{Q}_b'(0)\rangle}{w(b)} + F_a(t) \tag{A1}$$

with

$$v_{\mu\mathbf{r}}(a) \equiv \langle M_0^+(z)A_{\mu\mathbf{r}}(z); a\rangle \sum_\nu \int \frac{d\mathbf{r}'}{\omega_h} \frac{\partial}{\partial a_{\nu\mathbf{r}'}} \iint d\mathbf{p}\, d\mathbf{p}'\, g_\mu(\mathbf{p}) g_\nu(\mathbf{p}') \times \langle E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(F^e + z); a\rangle \tag{A2}$$

$$\hat{Q}_a(t) \equiv \exp[t(1-P)M^+](1-P)M^+\, \delta(A(z) - a) \tag{A3}$$

$$\hat{Q}_b'(0) \equiv \hat{Q}_b(0) - 2\sum_\mu \int \frac{d\mathbf{r}}{\omega_h} \iiint d\mathbf{p}\, d\mathbf{p}'\, d\mathbf{r}' \times \frac{\partial}{\partial b_{\mu\mathbf{r}}} \delta(A(z) - b) g_\mu(\mathbf{p}) \times \left(\frac{1}{\omega}\frac{\partial}{\partial z_{\mathbf{p}'\mathbf{r}'}} - \frac{z_{\mathbf{p}'\mathbf{r}'}}{F^e_{\mathbf{p}'}}\right) E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(F^e + z) \tag{A4}$$

$$F_a(t) \equiv Q_a(t) + \int dz\, \delta(A(z) - a) G_z(t) + \int_0^t ds \int dz\, \hat{Q}_a(t-s) G_z(s) \tag{A5}$$

where

$$M_0(z) = -\iint \frac{d\mathbf{p}\, d\mathbf{r}}{\omega} \frac{\partial}{\partial z_{\mathbf{pr}}} \left(-\sum_\alpha \frac{p_\alpha}{m} \partial_\alpha z_{\mathbf{pr}} + D_{\mathbf{p}}{}^e z_{\mathbf{pr}}\right) \tag{A6}$$

$$M_1(z) = \iint \frac{d\mathbf{p}\, d\mathbf{r}}{\omega} \iint \frac{d\mathbf{p}'\, d\mathbf{r}'}{\omega} \frac{\partial}{\partial z_{\mathbf{pr}}} \frac{\partial}{\partial z_{\mathbf{p}'\mathbf{r}'}} E_{\mathbf{pr};\mathbf{p}'\mathbf{r}'}(F^e + z) \tag{A7}$$

and $M(z)\rho_e(z) = 0$ holds exactly.

Let us consider the diffusion coefficient $E_{\mathbf{pr};\mathbf{p'r'}}(F^e + z)$. Since the fluctuating part $Z_{\mathbf{pr}}(t)$ is small compared to $F_{\mathbf{p}}{}^e$, we may assume $E_{\mathbf{pr};\mathbf{p'r'}}(F^e + z) = E_{\mathbf{pr};\mathbf{p'r'}}(F^e)$. The diffusion coefficient in the equilibrium state $E_{\mathbf{pr};\mathbf{p'r'}}(F^e)$ is related to the collision term by the fluctuation-dissipation theorem of the second kind, and is written as[5]

$$E_{\mathbf{pr};\mathbf{p'r'}}(F^e) = -\Delta(\mathbf{r} - \mathbf{r}')[D_{\mathbf{p}}{}^e F_{\mathbf{p}}{}^e\, \delta(\mathbf{p} - \mathbf{p}')] \tag{A8}$$

Then (A7) leads to

$$M_1(z) = M_1{}^+(z) = -\iiint \frac{d\mathbf{p}\, d\mathbf{p}'\, d\mathbf{r}}{\omega^2}\, [D_{\mathbf{p}}{}^e F_{\mathbf{p}}{}^e\, \delta(\mathbf{p} - \mathbf{p}')]\, \frac{\partial}{\partial z_{\mathbf{pr}}} \frac{\partial}{\partial z_{\mathbf{p'r}}} \tag{A9}$$

Since it contains the second differential operator with respect to z, we obtain for any linear function of z, $B(z)$,

$$M_1{}^+(z) B(z) = 0 \tag{A10}$$

Next, let us consider the linear collision operator $D_{\mathbf{p}}{}^e$ in the equilibrium state, which takes the form

$$D^e_{\mathbf{p}_1} X(\mathbf{p}_1) = \int d\mathbf{p}_2\, g_{21} \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\varphi\, [F^e_{\mathbf{p}_1^*} X(\mathbf{p}_2{}^*) + F^e_{\mathbf{p}_2^*} X(\mathbf{p}_1{}^*) - F^e_{\mathbf{p}_1} X(\mathbf{p}_2) - F^e_{\mathbf{p}_2} X(\mathbf{p}_1)] \tag{A11}$$

where $X(\mathbf{p})$ is an arbitrary function of $\mathbf{p}$, $g_{21} \equiv |\mathbf{p}_2 - \mathbf{p}_1|/m$, ρ is the impact parameter, φ is the azimuth, and $\mathbf{p}_i{}^*$ is the momentum of particle i in the restituting collision. Let us introduce $\tilde{D}_{\mathbf{p}}{}^e$ by

$$\tilde{D}_{\mathbf{p}}{}^e \psi(\mathbf{p}) \equiv (1/F_{\mathbf{p}}{}^e) D_{\mathbf{p}}{}^e [F_{\mathbf{p}}{}^e \psi(\mathbf{p})] \tag{A12}$$

Since

$$F^e_{\mathbf{p}_1^*} F^e_{\mathbf{p}_2^*} = F^e_{\mathbf{p}_1} F^e_{\mathbf{p}_2}$$

Eq. (A11) leads to

$$\tilde{D}^e_{\mathbf{p}_1} \psi(\mathbf{p}_1) = \int d\mathbf{p}_2\, g_{21} \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\varphi\, F^e_{\mathbf{p}_2} [\psi(\mathbf{p}_2{}^*) + \psi(\mathbf{p}_1{}^*) - \psi(\mathbf{p}_2) - \psi(\mathbf{p}_1)] \tag{A13}$$

It turns out easily from (A13) that $\tilde{D}_{\mathbf{p}}{}^e$ satisfies the following two properties:

$$(\psi_1(\mathbf{p}), \tilde{D}_{\mathbf{p}}{}^e \psi_2(\mathbf{p})) = (\tilde{D}_{\mathbf{p}}{}^e \psi_1(\mathbf{p}), \psi_2(\mathbf{p})) \tag{A14}$$

$$\tilde{D}_{\mathbf{p}}{}^e g_\mu(\mathbf{p}) = 0 \tag{A15}$$

where the scalar product is defined by

$$(\psi_1(\mathbf{p}), \psi_2(\mathbf{p})) \equiv \int d\mathbf{p}\, \psi_1(\mathbf{p}) \psi_2(\mathbf{p}) F_{\mathbf{p}}{}^e \tag{A16}$$

Using $\tilde{D}_{\mathbf{p}}{}^{e}$ and (A14) and (A15), we obtain

$$M_1{}^+(z)\,\delta(A(z) - a) = 0 \tag{A17}$$

It turns out easily from (A10) and (A17) that the second term of $v_{\mu\mathbf{r}}(a)$ and the $M_1{}^+(z)$ in $\hat{Q}_a(t)$ disappear. Furthermore, the second term of $\hat{Q}_b{}'(0)$ also vanishes due to (A14) and (A15). Thus, the master equation (A1) leads to (54) and the reduced equations of motion (55). In equilibrium systems, the propagator is also given by the drift term $M_0{}^+(f)$, whereas the hydrodynamic fluctuating forces $R'_{\mu\mathbf{r}}$ have two parts.

APPENDIX B. DERIVATION OF (77)

First let us consider the collision term $C_{\mathbf{pr}}(f)$ in $M_0{}^+(f)$. When the deviation of $F_{\mathbf{pr}}(t)$ from the local equilibrium distribution $F^l_{\mathbf{pr}}(t)$ is small, the collision term can be linearized around $F^l_{\mathbf{pr}}(t)$:

$$C_{\mathbf{pr}}(F) \approx D_{\mathbf{pr}}\,\delta F_{\mathbf{pr}} \tag{B1}$$

The linear collision operator $D_{\mathbf{pr}}$ takes the form

$$\begin{aligned} D_{\mathbf{p}_1\mathbf{r}}X(\mathbf{p}_1) \equiv \int d\mathbf{p}_2\, g_{21} \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\varphi\, [F^l_{\mathbf{p}_1{}^*\mathbf{r}}X(\mathbf{p}_2{}^*) + F^l_{\mathbf{p}_2{}^*\mathbf{r}}X(\mathbf{p}_1{}^*) \\ - F^l_{\mathbf{p}_1\mathbf{r}}X(\mathbf{p}_2) - F^l_{\mathbf{p}_2\mathbf{r}}X(\mathbf{p}_1)] \end{aligned} \tag{B2}$$

Since $F^l_{\mathbf{p}_1{}^*\mathbf{r}}F^l_{\mathbf{p}_2{}^*\mathbf{r}} = F^l_{\mathbf{p}_1\mathbf{r}}F^l_{\mathbf{r}_2\mathbf{r}}$, we can introduce the operator $\tilde{D}_{\mathbf{pr}}$ defined by (78) analogously to (A12). Therefore, the following property for $\tilde{D}_{\mathbf{pr}}$ can be derived in the same manner as in Appendix A:

$$\int d\mathbf{p}\, \psi_1(\mathbf{p})[\tilde{D}_{\mathbf{pr}}\psi_2(\mathbf{p})]F^l_{\mathbf{pr}} = \int d\mathbf{p}\, [\tilde{D}_{\mathbf{pr}}\psi_1(\mathbf{p})]\psi_2(\mathbf{p})F^l_{\mathbf{pr}} \tag{B3}$$

The length cutoff b_h is taken to be much longer than the mean free path l_f. The b_h, however, must be shorter than the length scale l_h of the hydrodynamic modes, in which a large deviation from thermal equilibrium is produced. Then the term including the gradient in $M_0{}^+$ is negligible in (76), and the propagator reduces to

$$U(t) \approx \exp\left[t(1 - P)(1/\omega) \iint d\mathbf{p}\, d\mathbf{r}\, \{D_{\mathbf{pr}}\,\delta f_{\mathbf{pr}}\}(\partial/\partial f_{\mathbf{pr}})\right] \tag{B4}$$

Introducing $\tilde{\delta} f_{\mathbf{pr}} \equiv (1/F^l_{\mathbf{pr}})\,\delta f_{\mathbf{pr}}$, and using $\tilde{D}_{\mathbf{pr}}$ and its property (B3), we can write (66) as (77).

APPENDIX C. DERIVATION OF (82), (83), AND THE FLUCTUATING HYDRODYNAMIC EQUATIONS

Let us consider the linear coefficients $L_{\mu\alpha;\nu\beta}$. Inserting (67) into (81), we obtain

$$L_{\mu\alpha;\nu\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_{1\gamma,\alpha;1\delta,\beta} & L_{1\gamma,\alpha;2\beta} + \sum_{\delta} u_\delta L_{1\gamma,\alpha;1\delta,\beta} \\ 0 & L_{2\alpha;1\delta,\beta} + \sum_{\gamma} u_\gamma L_{1\gamma,\alpha;1\delta,\beta} & L_{2\alpha;2\beta} + \sum_{\gamma} u_\gamma (L_{1\gamma,\alpha;2\beta} + L_{2\alpha;1\gamma,\beta}) \\ & & + \sum_{\gamma}\sum_{\delta} u_\gamma u_\delta L_{1\gamma,\alpha;1\delta,\beta} \end{pmatrix} \tag{C1}$$

with

$$L_{1\gamma,\alpha;1\delta,\beta} \equiv (1/k_B) \int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^e)\, J_{1\mathbf{r}}^{\alpha\gamma}(\mathbf{p})] J_{1\mathbf{r}}^{\beta\delta}(\mathbf{p}) F_{\mathbf{p}}{}^e \tag{C2}$$

$$L_{1\gamma,\alpha;2\beta} \equiv (1/k_B) \int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^e)\, J_{1\mathbf{r}}^{\alpha\gamma}(\mathbf{p})] J_{2\mathbf{r}}^{\beta} F_{\mathbf{p}}{}^e \tag{C3}$$

$$L_{2\alpha;2\beta} \equiv (1/k_B) \int_0^\infty dt \int d\mathbf{p}\, [\exp(t\tilde{D}_{\mathbf{p}}{}^e) J_{2\mathbf{r}}^{\alpha}(\mathbf{p})] J_{2\mathbf{r}}^{\beta}(\mathbf{p}) F_{\mathbf{p}}{}^e \tag{C4}$$

Taking an isotropy and symmetry into account, we have

$$L_{1\gamma,\alpha;2\beta} = 0 \tag{C5}$$

$$L_{1\gamma,\alpha;1\delta,\beta} = T[\eta(\delta_{\gamma\delta}\,\delta_{\alpha\beta} + \delta_{\gamma\beta}\,\delta_{\alpha\delta}) + \eta'\,\delta_{\gamma\delta}\,\delta_{\alpha\beta}] \tag{C6}$$

$$L_{2\alpha;2\beta} = T^2\kappa\delta_{\alpha\beta} \tag{C7}$$

Next let us consider the thermodynamic forces. As was shown by Green,[13] the following relations hold for the entropy $\sigma(a)$:

$$(1/\omega_h)(\partial/\partial a_{1\alpha,\mathbf{r}})\sigma(a) = -u_\alpha/T \tag{C8}$$

$$(1/\omega_h)(\partial/\partial a_{2\mathbf{r}})\sigma(a) = 1/T \tag{C9}$$

Then the thermodynamic forces $X_{\mu\mathbf{r}}^{\alpha}$ for $\mu = (1, \beta)$ and $\mu = 2$ are written as

$$X_{1\mathbf{r}}^{\alpha\beta} = -[(1/T)\,\partial_\beta u_\alpha + u_\alpha X_{2\mathbf{r}}^{\beta}] \tag{C10}$$

$$X_{2\mathbf{r}}^{\alpha} = -(1/T^2)\,\partial_\alpha T \tag{C11}$$

Therefore, the dissipative terms $\sum_\nu \sum_\delta X_{\nu\mathbf{r}}^{\delta} L_{\mu\alpha;\nu\delta}$ for $\mu = (1, \beta)$ and $\mu = 2$ lead to the viscous-stress tensor $P'_{\alpha\beta,\mathbf{r}}$ and the heat flow $q'_{\beta,\mathbf{r}}$, respectively:

$$\sum_\nu \sum_\delta X_{\nu\mathbf{r}}^{\delta} L_{1\beta,\alpha;\nu\delta} \equiv P'_{\alpha\beta,\mathbf{r}} = 2\eta\Big(Y_{1\mathbf{r}}^{\alpha\beta} - \tfrac{1}{3}\sum_\delta Y_{1\mathbf{r}}^{\delta\delta}\,\delta_{\alpha\beta}\Big) + \zeta \sum_\delta Y_{1\mathbf{r}}^{\delta\delta}\,\delta_{\alpha\beta} \tag{C12}$$

$$\sum_\nu \sum_\delta X_{\nu\mathbf{r}}^{\delta} L_{2\alpha;\nu\delta} = q'_{\alpha,\mathbf{r}} + \sum_\beta u_\beta P'_{\alpha\beta,\mathbf{r}} \tag{C13}$$

where $\zeta \equiv (2/3)\eta + \eta'$,

$$Y_{1\mathbf{r}}^{\alpha\beta} \equiv -(1/2)(\partial_\beta u_\alpha + \partial_\alpha u_\beta) \tag{C14}$$

$$q'_{\alpha,\mathbf{r}} \equiv -\kappa\, \partial_\alpha T \tag{C15}$$

Equation (C15) represents the Fourier law. For the coefficients η and κ, we obtain (82) and (83) from (C6) and (C7).

On the other hand, the fluctuating forces $S_{\mu\mathbf{r}}^{\alpha}(t)$ take the form

$$S_{\mu\mathbf{r}}^{\alpha}(t) = \begin{pmatrix} 0 \\ S_{1\mathbf{r}}^{\alpha\beta}(t) \\ S_{2\mathbf{r}}^{\alpha}(t) + \sum_\beta u_\beta S_{1\mathbf{r}}^{\alpha\beta}(t) \end{pmatrix} \tag{C16}$$

with

$$S_{1\mathbf{r}}^{\alpha\beta}(t) \equiv \left\{\int d\mathbf{p}\, [\exp(t\tilde{D}_\mathbf{p}{}^e)\, J_{1\mathbf{r}}^{\alpha\beta}(\mathbf{p})]\, \delta f_{\mathbf{pr}}\right\}_{f\to F(0)} \tag{C17}$$

$$S_{2\mathbf{r}}^{\alpha}(t) \equiv \left\{\int d\mathbf{p}\, [\exp(t\tilde{D}_\mathbf{p}{}^e)\, J_{2\mathbf{r}}^{\alpha}(\mathbf{p})]\, \delta f_{\mathbf{pr}}\right\}_{f\to F(0)} \tag{C18}$$

Therefore, we can transform the conservation equations (70) with (71) and (81) into

$$\partial_t n = -\nabla\cdot[n\mathbf{u}] \tag{C19}$$

$$mn[\partial_t + \mathbf{u}\cdot\nabla]u_\alpha = -\sum_\beta \partial_\beta P_{\alpha\beta,\mathbf{r}} - \sum_\beta \partial_\beta S_{1\mathbf{r}}^{\alpha\beta}(t) \tag{C20}$$

$$(3/2)nk_B[\partial_t + \mathbf{u}\cdot\nabla]T = -\sum_\alpha \partial_\alpha q'_{\alpha,\mathbf{r}} - \sum_\alpha\sum_\beta [P_{\alpha\beta,\mathbf{r}} + S_{1\mathbf{r}}^{\alpha\beta}(t)]\,\partial_\alpha u_\beta$$
$$- \sum_\alpha \partial_\alpha S_{2\mathbf{r}}^{\alpha}(t) \tag{C21}$$

where $P_{\alpha\beta,\mathbf{r}}$ is the pressure tensor $P_{\alpha\beta,\mathbf{r}} \equiv \delta_{\alpha\beta}P_\mathbf{r}{}^0 + P'_{\alpha\beta,\mathbf{r}}$, with $P_\mathbf{r}{}^0$ the thermodynamic pressure $P_\mathbf{r}{}^0 = nk_BT$. Equations (C19)–(C21) agree with the fluctuating hydrodynamic equations proposed by Landau and Lifshitz.

APPENDIX D. AN L-TYPE REDUCTION FOR DERIVING (86)–(89)

Let us introduce $\tilde{Z}_{\mathbf{pr}}(t)$ by

$$\tilde{Z}_{\mathbf{pr}}(t) \equiv (1/F_\mathbf{p}{}^e)Z_{\mathbf{pr}}(t) \tag{D1}$$

Then the linear Boltzmann–Langevin equation (4) and eq. (84) can be transformed into

$$\partial_t \tilde{Z}_{\mathbf{pr}}(t) = \tilde{L}_{\mathbf{pr}} \tilde{Z}_{\mathbf{pr}}(t) + \tilde{G}_{\mathbf{pr}}(t) \tag{D2}$$

$$\tilde{Z}_{\mathbf{pr}}(t) = \exp(t\tilde{L}_{\mathbf{pr}})\, \tilde{Z}_{\mathbf{pr}}(0) + \int_0^t ds \exp[(t-s)\tilde{L}_{\mathbf{pr}}]\, \tilde{G}_{\mathbf{pr}}(s) \tag{D3}$$

with

$$\tilde{L}_{\mathbf{pr}} \equiv -\sum_\alpha (p_\alpha/m)\, \partial_\alpha + \tilde{D}_{\mathbf{p}}{}^e \tag{D4}$$

$$\tilde{G}_{\mathbf{pr}}(t) \equiv (1/F_{\mathbf{p}}{}^e) G_{\mathbf{pr}}(t) \tag{D5}$$

Analogously to the discussion in Section 3, we have the conservation equations (70) with the flux densities

$$j^\alpha_{\mu\mathbf{r}}(t) = ch^\alpha_{\mu\mathbf{r}}(t) + \mathscr{T}^\alpha_{\mu\mathbf{r}}(t) \tag{D6}$$

with

$$h^\alpha_{\mu\mathbf{r}}(t) \equiv (1/c) \int d\mathbf{p}\, (p_\alpha/m) g_\mu(\mathbf{p}) \{F_{\mathbf{p}}{}^e + [P\tilde{Z}_{\mathbf{pr}}(t)] F_{\mathbf{p}}{}^e\} \tag{D7}$$

$$\mathscr{T}^\alpha_{\mu\mathbf{r}}(t) \equiv \int d\mathbf{p}\, (p_\alpha/m) g_\mu(\mathbf{p}) [(1-P)\tilde{Z}_{\mathbf{pr}}(t)] F_{\mathbf{p}}{}^e \tag{D8}$$

where c is the mean particle density. Inserting (D3) into (D8) and using the operator identity

$$e^{tL} = Pe^{tL} + \int_0^t ds\, (1-P) e^{s(1-P)L} (1-P) L P e^{(t-s)L} + e^{t(1-P)L}(1-P) \tag{D9}$$

we obtain

$$\begin{aligned} \mathscr{T}^\alpha_{\mu\mathbf{r}}(t) = \int d\mathbf{p}\, (p_\alpha/m) g_\mu(\mathbf{p}) \int_0^t ds\, [(1-P)\{\exp[s(1-P)\tilde{L}_{\mathbf{pr}}]\} \\ \times (1-P)\tilde{L}_{\mathbf{pr}} P \tilde{Z}_{\mathbf{pr}}(t-s)] F_{\mathbf{p}}{}^e + T^\alpha_{\mu\mathbf{r}}(t) \end{aligned} \tag{D10}$$

Here the fluctuating forces T consist of two parts, T_1, derived from the first term of (D3), and T_2, derived from the second term:

$$T^\alpha_{1\mu\mathbf{r}}(t) \equiv \int d\mathbf{p}\, (p_\alpha/m) g_\mu(\mathbf{p}) [\{\exp[t(1-P)\tilde{L}_{\mathbf{pr}}]\}(1-P)\tilde{Z}_{\mathbf{pr}}(0)] F_{\mathbf{p}}{}^e \tag{D11}$$

$$\begin{aligned} T^\alpha_{2\mu\mathbf{r}}(t) \equiv \int d\mathbf{p}\, (p_\alpha/m) g_\mu(\mathbf{p}) \int_0^t ds \\ \times [\{\exp[(t-s)(1-P)\tilde{L}_{\mathbf{pr}}]\}(1-P)\tilde{G}_{\mathbf{pr}}(s)] F_{\mathbf{p}}{}^e \end{aligned} \tag{D12}$$

Next let us take as P the following L-type projector onto the collisional invariants $g(\mathbf{p}) \equiv \{g_\mu(\mathbf{p})\}$:

$$P\psi(\mathbf{p}) = \sum_\lambda \sum_\nu (\psi(\mathbf{p}), g_\lambda(\mathbf{p}))(g(\mathbf{p}), g(\mathbf{p}))_{\lambda\nu}^{-1} g_\nu(\mathbf{p}) \tag{D13}$$

where the scalar product is defined by (A16). It turns out easily that the projector (D13) has the following properties:

$$(\psi_1(\mathbf{p}), P\psi_2(\mathbf{p})) = (P\psi_1(\mathbf{p}), \psi_2(\mathbf{p})) \tag{D14}$$

$$P\tilde{Z}_{\mathbf{pr}}(t) = \sum_\lambda \sum_\nu \delta A_{\lambda\mathbf{r}}(t)(g(\mathbf{p}), g(\mathbf{p}))_{\lambda\nu}^{-1} g_\nu(\mathbf{p}) \tag{D15}$$

$$P[(p_\alpha/m)g_\mu(\mathbf{p})] = \begin{pmatrix} p_\alpha/m \\ \delta_{\alpha\beta}p^2/3m \\ (5/2)k_B T_e p_\alpha/m \end{pmatrix} \tag{D16}$$

where T_e denotes the equilibrium temperature and

$$\delta A_{\lambda\mathbf{r}}(t) \equiv \int g_\lambda(\mathbf{p}) Z_{\mathbf{pr}}(t)\, d\mathbf{p} \tag{D17}$$

Therefore, (D10) leads to

$$\mathscr{T}_{\mu\mathbf{r}}^\alpha(t) = \int_0^t ds\, (1/k_B) \times \sum_\nu \sum_\beta (J_\mu{}^\alpha(\mathbf{p}), \{\exp[s(1-P)\tilde{L}_{\mathbf{pr}}]\} J_\nu{}^\beta(\mathbf{p}))\chi_{\nu\mathbf{r}}^\beta(t) + T_{\mu\mathbf{r}}^\alpha(t) \tag{D18}$$

where

$$J_\mu{}^\alpha(\mathbf{p}) \equiv (1-P)[(p_\alpha/m)g_\mu(p)] = \begin{pmatrix} 0 \\ (p_\alpha/m)p_\beta - \delta_{\alpha\beta}p^2/3m \\ (p_\alpha/m)[p^2/2m - \frac{5}{2}k_B T_e] \end{pmatrix} \tag{D19}$$

$$\chi_{\nu\mathbf{r}}^\alpha(t) \equiv \sum_\lambda k_B (g(p), g(p))_{\nu\lambda}^{-1}[-\partial_\alpha \delta A_{\lambda\mathbf{r}}(t)] \tag{D20}$$

For the fluctuating forces, we have

$$T_{1\mu\mathbf{r}}^\alpha(t) = (J_\mu{}^\alpha(\mathbf{p}), \{\exp[t(1-P)\tilde{L}_{\mathbf{pr}}]\}(1-P)\tilde{Z}_{\mathbf{pr}}(0)) \tag{D21}$$

$$T_{2\mu\mathbf{r}}^\alpha(t) = \int_0^t ds\, (J_\mu{}^\alpha(\mathbf{p}), \{\exp[(t-s)(1-P)\tilde{L}_{\mathbf{pr}}]\}(1-P)\tilde{G}_{\mathbf{pr}}(s)) \tag{D22}$$

Applying the hydrodynamic space-time coarse-graining and using (A14), we obtain (86)–(89). They agree with the results derived with the aid of the reduction method formulated in Section 2.

NOTE ADDED IN PROOF

It should be noted that (77) is valid only for $S^{\alpha}_{\mu\mathbf{r}}(t)$ of (76) and cannot be used for the fluctuating forces $S^{\alpha}_{\mu\mathbf{r}}(t)$ of (72) and (80). Due to the streaming term of $M_0{}^+$ involved in $U(t)$, (66) does not vanish as far as the spatial inhomogeneity is retained, whereas (77) vanishes in a time of the order of the mean free time τ_f. As the complete equilibrium is attained, the fluctuating force $S^{\alpha}_{\mu\mathbf{r}}(t)$ vanishes and the molecular fluctuating force $G_{\mathbf{pr}}(t)$ becomes important.

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